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LETTER TO THE EDITOR

A note on a generalisation of Weyl's theory of gravitation

T Dereli† and R W Tucker
Department of Physics, University of Lancaster, Lancaster, UK

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Abstract. A scale-invariant gravitational theory due to Bach and Weyl is generalised by the inclusion of space–time torsion. The difference between the arbitrary and zero torsion constrained variations of the Weyl action is elucidated. Conformal rescaling properties of the gravitational fields are discussed. A new class of classical solutions with torsion is presented.

A scale-invariant theory of gravitation usually associated with the names of Bach and Weyl is based on the action

$$I = \int C_{ab} \wedge * C^{ab} \tag{1}$$

where $C_{ab} = -C_{ba}$ are the six independent Weyl curvature two-forms and $*$ denotes the Hodge duality operation with respect to a Lorentz signatured metric‡. The integral is over the space–time manifold. We recently became aware of an article by Fiedler and Schmming (1980), dedicated to the study of classical solutions of the theory described by (1), where a detailed list of references can also be found. Originally Bach considered a metric compatible, torsion-free connection and the same approach is adopted by Fiedler and Schmming. Then the field equations derived from metric component variations of (1) turn out to be fourth-order partial differential equations in the natural components of the metric tensor. Our purpose in this note is to discuss a simple generalisation of Bach–Weyl theory which results when the assumption of a torsion-free connection in (1) is relaxed. This is a natural generalisation in the light of a gauge approach to gravitation (Benn *et al* 1980, 1982). Some new classical solutions that result from a modified double dual curvature condition are pointed out.

We consider action (1) written in terms of a metric and a metric compatible but otherwise arbitrary connection. Then independent frame and connection variations of (1) yield the field equations

$$P_b \wedge * C^b{}_a + i_a C_{bc} \wedge * C^{bc} - C_{bc} \wedge i_a * C^{bc} = 0, \tag{2}$$

$$D * C^a{}_b = 0, \tag{3}$$

respectively. Under the assumption of vanishing torsion, $T^a = 0$, these reduce to

$$\mathring{P}_b \wedge * \mathring{C}^b{}_a = 0, \tag{4}$$

$$\mathring{D} * \mathring{C}^a{}_b = 0, \tag{5}$$

† Permanent address: Physics Department, Middle East Technical University, Ankara, Turkey.
‡ We are using the formalism of exterior differential forms. Basic definitions and conventions are presented separately in an Appendix.

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where zeros indicate the use of the Christoffel connection. However, equations (4) and (5) are not equivalent to the fourth-order Bach–Weyl equation studied by Fiedler and Schmming (1980). In order to obtain the Bach–Weyl equation from a consistent variational principle, we impose the zero torsion condition by introducing Lagrange multiplier forms in the action. Thus instead of (1) we consider the SO(1, 3) invariant action

$$I_1 = \int [C_{ab} \wedge * C^{ab} + \lambda_a \wedge (\mathrm{d}e^a + \omega^a{}_b \wedge e^b)] \tag{6}$$

where λ_a, $a = 0, 1, 2, 3$, are four independent Lagrange multiplier two-forms. Independent frame, connection and λ_a variations of (6) give

$$P_b * C^b{}_a + i_a C_{bc} \wedge * C^{bc} - C_{bc} \wedge i_a * C^{bc} + D\lambda_a = 0, \tag{7}$$

$$2D * C_{ab} + \tfrac{1}{2}(\lambda_a \wedge e_b - \lambda_b \wedge e_a) = 0, \tag{8}$$

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0, \tag{9}$$

respectively. Equation (9) imposes the constraint $T^a = 0$. Equations (7) and (8) are now to be solved under this constraint. It is possible to solve (8) explicitly for λ_a:

$$\lambda_a = 2i_b(\mathring{D} * \mathring{C}^b{}_a) + *(e^b \wedge e^c) i_a * (\mathring{D} * \mathring{C}_{bc}). \tag{10}$$

Then (10) is substituted into (7). The resulting expression,

$$\mathring{P}_b \wedge * \mathring{C}^b{}_a + 2\mathring{D}(i_b(\mathring{D} * \mathring{C}^b{}_a)) + *(e^b \wedge e^c) \wedge \mathring{D}(i_a * (\mathring{D} * \mathring{C}_{bc})) = 0, \tag{11}$$

is known as the Bach–Weyl equation†. It is now clear that any torsion-free solution of (2) and (3) is also a solution of (11). But the converse need not be true in general. We regard (2) and (3) as a convenient generalisation of the Bach–Weyl equation in the presence of torsion. This point will be appreciated more if spinor matter couplings to the theory are examined.

It is worth pointing out at this stage the conformal rescaling properties of the gravitational fields in the generalisation we are discussing. The original Bach–Weyl theory is invariant under local scalings of the metric, since the Weyl curvature two-forms written in terms of Christoffel connection are scale invariant and the action density four-form is constructed out of Weyl curvature two-forms and their Hodge duals. In the presence of arbitrary metric compatible connection, the corresponding Weyl curvature two-forms and hence the action (1) is still locally scale invariant provided the same SO(3, 1) connection is adopted for all conformally related metrics. That is to say, if

$$e^a \to e^{\lambda(x)} e^a \tag{12}$$

then we require

$$\omega^a{}_b \to \omega^a{}_b. \tag{13}$$

It follows from these scaling rules and the structure equations that

$$T^a \to e^{\lambda(x)}(T^a + \mathrm{d}\lambda \wedge e^a), \tag{14}$$

$$R^a{}_b \to R^a{}_b. \tag{15}$$

† Equation (11) can be brought into the form given by Fiedler and Schmming (1980) with the aid of Bianchi identities and the symmetry properties of the components of $\mathring{C}_{ab}$.

Therefore the torsion of any particular connection can be changed by adopting a new metric in which the conformally related frames are orthonormal. It is possible to construct many other locally scale-invariant actions in our approach, some of which have already been investigated by Dereli and Tucker (1981). The above arguments concerning the difference between arbitrary and zero torsion constrained variations are equally applicable to these theories. It is well known that both types of variations lead to the same set of field equations for Einstein–Cartan theory which is described by an action linear in the curvatures. The distinction between the two types of variations appears to be significant whenever a theory (e.g. one which is described by an action quadratic in curvatures and/or torsions) admits 'dynamical' torsion, i.e. where T^a is not related to other degrees of freedom algebraically.

Finally we would like to discuss some new classical solutions of the Bach–Weyl theory. But before that, two observations are in order. First, we note that any torsion-free geometry described by a vacuum Einstein metric constitutes also a solution of equations (2) and (3). This follows from the vacuum Einstein equations $\mathring{P}_a = 0$ and the double duality of the corresponding curvature two-forms. Next, all geometries conformally related to vacuum Einstein solutions are again solutions of (2) and (3). This is due to the scale invariance of action (1). In order to illustrate our discussion we now specialise to static, spherically symmetric field configurations. The unique vacuum Einstein solution described by the Schwarzschild metric together with zero torsion and all geometries conformally related to it are also solutions of (2) and (3). According to the scaling rules (12)–(15) this class of solutions, which can be characterised by $P_a = 0$ and $C_{ab} \neq 0$, may well include some geometries with non-vanishing torsion. A distinct class of solutions can be constructed by considering the expression

$$* (R_{ab} - \tfrac{1}{2}\lambda e_a \wedge e_b) = -\tfrac{1}{2}\varepsilon_{ab}{}^{cd}(R_{cd} - \tfrac{1}{2}\lambda e_c \wedge e_d) \tag{16}$$

where λ is an arbitrary constant and ε_{abcd} is the totally antisymmetric object with $\varepsilon_{0123} = 1$. Any geometry that satisfies the modified double dual curvature condition (16) may be characterised by the condition (1) $Q = 2\lambda$ so that $P_a \neq 0$ and (2) $C_{ab} = 0$ provided they also satisfy $e^a \wedge DT_a = 0$ (Benn *et al* 1981). Therefore these geometries together with all conformally related geometries form an independent class of solutions of (2) and (3). The spaces of constant curvature defined by

$$R_{ab} = \tfrac{1}{2}\lambda e_a \wedge e_b \tag{17}$$

constitute a subclass of solutions. Any torsion-free geometry that solves (16) is conformally flat. The Schwarzschild metric with a 'cosmological constant k',

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{kr^2}{3}\right) dt^2 + \frac{dr^2}{(1 - 2m/r + \frac{1}{3}kr^2)} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2), \tag{18}$$

together with the torsion two-forms

$$T^0 = \frac{2(\frac{1}{3}k + \lambda)r + (2m/r^2)}{(1 - 2m/r + \frac{1}{3}kr^2)^{1/2}} \, dr \wedge dt, \qquad T^k = 0, \qquad k = 1, 2, 3, \tag{19}$$

also solves (16). We have established the existence of at least two distinct classes of classical solutions to equations (2) and (3). The torsion-free solutions in both classes also satisfy the Bach–Weyl equation (11). However, both classes include solutions with non-vanishing torsion. Thus we are able to conclude that there is no Birkhoff-type theorem for the gravitational theory based solely on action (1). Such distinct solutions must presumably be distinguished by topological considerations.

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Appendix

Space-time is a four-dimensional differentiable manifold with a Lorentz signatured metric

$$g = \eta_{ab} e^a \otimes e^b \tag{A1}$$

where $\eta_{ab} = \mathrm{diag}\,(-+++)$ and e^a, $a = 0, 1, 2, 3$, are the orthonormal basis one-forms. An independent but metric compatible connection whose SO(1, 3) components are denoted by $\omega_{ab} = -\omega_{ba}$ will be used. The structure equations

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = T^a, \tag{A2}$$

$$\mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R^a{}_b, \tag{A3}$$

define, respectively, the torsion two-forms

$$T^a = T_{bc,}{}^a e^b \wedge e^c \tag{A4}$$

and the curvature two-forms

$$R^a{}_b = \tfrac{1}{2} R_{cd,}{}^a{}_b e^c \wedge e^d. \tag{A5}$$

The Ricci one-forms

$$P_a = i_b R^b{}_a \tag{A6}$$

where the interior operators i_a, $a = 0, 1, 2, 3$, are defined by $i_a e^b = \delta_a{}^b$. The curvature scalar

$$Q = i_a P^a. \tag{A7}$$

Then the Weyl curvature two-forms

$$C_{ab} = R_{ab} - \tfrac{1}{2}(e_a \wedge P_b - e_b \wedge P_a) + \tfrac{1}{6} e_a \wedge e_b Q. \tag{A8}$$

It is easily verified that

$$i_a C^a{}_b \equiv * (e^a \wedge * C_{ab}) = 0. \tag{A9}$$

A metric compatible connection with torsion may be uniquely decomposed according to

$$\omega_{ab} = \mathring{\omega}_{ab} + K_{ab} \tag{A10}$$

where the Christoffel connection one-forms are determined from

$$\mathrm{d}e^a + \mathring{\omega}^a{}_b \wedge e^b = 0 \tag{A11}$$

and the equations

$$K^a{}_b \wedge e^b = T^a \tag{A12}$$

determine the contortion one-forms $K_{ab} = -K_{ba}$. Then we define the Christoffel curvature two-forms

$$\mathring{R}^a{}_b = \mathrm{d}\mathring{\omega}^a{}_b + \mathring{\omega}^a{}_c \wedge \mathring{\omega}^c{}_b \tag{A13}$$

and the corresponding $\mathring{P}_a$, $\mathring{Q}$ and $\mathring{C}_{ab}$ associated with the Christoffel connection.

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